

Landau damping in dilute Bose gases

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Landau damping in weakly interacting Bose gases is investigated by means of perturbation theory. Our approach points out the crucial role played by Bose-Einstein condensation and yields an explicit expression for the decay rate of elementary excitations in both uniform and non uniform gases. Systematic results are derived for the phonon width in homogeneous gases interacting with repulsive forces. Special attention is given to the low and high temperature regimes.

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The collective excitations of gases of alkali atoms confined in magnetic traps have been the object of recent experimental measurements [1]. The observed frequencies very well agree with the theoretical predictions of mean field theory [2], confirming the validity of the Bogoliubov scheme, later extended by Gross and Pitaevskii to the non homogeneous case, for describing the dynamic behaviour of dilute Bose gases. Conversely, the damping mechanisms associated with such excitations are not well understood and still represent a challenging question for theoretical investigation.

The damping of collective modes can have various origins. At $T = 0$ it can arise because of decay into two or more excitations with lower energy. This mechanism, which is well understood in uniform Bose superfluids [3], is not active for the lowest modes of a trapped gas, because of discretization of levels. Damping can be due to other non linear phenomena of classical or quantum nature [4]. Experiments with magnetic traps however point out the occurrence of damping in an almost linear regime where such effects should be negligible. At finite temperature damping can be due to collisional effects of dissipative type. These effects are expected to be important at high temperature and density. Another mechanism, holding in the collisionless regime, is Landau damping which occurs when the collective excitations can decay due to coupling with transitions associated with other elementary excitations and occurring at the same frequency. Landau damping is not associated with thermalization processes and can be well described in the framework of mean field theory (see, for example, [5] §§28-29, 34 and [6]). In a Bose gas it arises only at finite temperature. Its possible relevance to explain the experimental data in trapped Bose gases has been recently proposed by Liu and Schieve [7].

The purpose of this letter is to develop the microscopic formalism of Landau damping in a dilute Bose gas. The aim is twofold. On the one hand we provide a systematic description of Landau damping in homogeneous systems covering the regimes of low temperature, first investigated by Hohenberg and Martin [8], and the one at higher temperatures first investigated by Szepfalusy and Kondor [9]. On the other hand we provide a formalism suitable for the calculation of Landau damping in trapped gases where the classification of elementary excitations significantly differs from the one of uniform systems. Our formalism is based on perturbation theory and points out the crucial role played by Bose-Einstein condensation. Previous approaches were based on kinetic theory for superfluids [5] or on the use of Green's function techniques [8].

Let E be the energy of the system associated with the occurrence of a classical oscillation of frequency ω induced, for example, by some external drive as happens in the experiments of ref. [1]. By classical oscillation we mean that the number of quanta of oscillation is very large. Due to interaction effects, the thermal component of the gas can either absorb or emit quanta of this oscillation, thereby giving rise to the following expression for the energy loss

$$\dot{E}_{os} = -\hbar\omega(W^{(a)} - W^{(e)}) . \quad (1)$$

In the above equation $W^{(a)}$ and $W^{(e)}$ are, respectively, the probabilities of absorption and emission of the corresponding quantum $\hbar\omega$ given, in perturbation theory, by

$$W = \pi \sum_{i,k} | \langle k | V_{int} | i \rangle |^2 , \quad (2)$$

where the matrix element is associated with a transition in which the i -th excitation, available in the system because of thermal activation, is transformed into the k -th one and $E_k = E_i + \hbar\omega$ in the case of absorption (a), and $E_k = E_i - \hbar\omega$ in the case of emission (e).

In second quantization the interaction term can be written in the following way

$$V_{int} = \frac{g}{2} \int d\mathbf{r} \, \psi^\dagger \psi^\dagger \psi \psi , \quad (3)$$

where the coupling constant g is related to the s -wave scattering length through the usual formula $g =$

$4\pi\hbar^2 a/m$. The field operator ψ can be written as $\psi = \psi_0 + \delta\psi$ where ψ_0 is the order parameter at equilibrium while $\delta\psi$ characterizes its fluctuations which can be expressed in terms of the annihilation (α) and creation (α^\dagger) operators relative to the elementary modes of the system in the usual form

$$\delta\psi = \sum_j \left(u_j(\mathbf{r})\alpha_j + v_j^*(\mathbf{r})\alpha_j^\dagger \right) \quad (4)$$

and u, v can be determined, for example, using Bogoliubov theory or its extensions to finite temperature [10,11]. In the sum (4) one should distinguish between the collective excitation whose decay is the object of the present work and for which we use the notation $u_{osc}, v_{osc}, \alpha_{osc}, \alpha_{osc}^\dagger$, and the other elementary modes which are thermally excited and for which we will use the indices i, k already employed in (2). By keeping in (3) only terms linear in $\alpha_{osc}, (\alpha_{osc}^\dagger)$ and in the product $\alpha_k^\dagger \alpha_i$ ($\alpha_k \alpha_i^\dagger$), we investigate the desired process where a quantum of oscillation $\hbar\omega$ is annihilated (created) and the i -th excitation is transformed into the k -th one. The same formalism permits to investigate also other decay processes where both the excitations i and k are created (Beliaev decay of a phonon into two phonons [3]) and $E_k + E_i = \hbar\omega$. In our case we find the following result for the rate of energy loss:

$$\dot{E} = -\omega 2\pi \sum_{ik} |A_{ki}|^2 \delta(E_k - E_i - \hbar\omega) (f_k - f_i). \quad (5)$$

where $E = \hbar\omega n_{osc}$ is the energy of the classical oscillation ($n_{osc} \gg 1$) and

$$A_{ki} = 2g \int d\mathbf{r} \psi_0 [(u_k^* v_i + v_k^* v_i + u_k^* u_i) u_{osc} + (v_k^* u_i + v_k^* v_i + u_k^* u_i) v_{osc}] \quad (6)$$

is the relevant matrix element for the process. In deriving (5) we have assumed that at equilibrium the states i, k are thermally occupied with the usual Bose factor $f_j = [\exp(\beta E_j) - 1]^{-1}$. Introducing the damping rate through the relation $\dot{E} = -2\gamma E$ we finally obtain the relevant formula

$$\gamma = -\omega\pi \sum_{ik} |A_{ki}|^2 \delta(E_k - E_i - \hbar\omega) \frac{\partial f(E_i)}{\partial E}, \quad (7)$$

where we have further assumed $\hbar\omega \ll T$. Eqs. (6,7) can be further simplified by calculating the relevant matrix elements in semiclassical approximation and can be used to calculate the damping rate of collective excitations in a trapped Bose gas. This calculation will be the object of a future work. In the following we will use them for a systematic discussion of Landau damping in a homogeneous gas where all the ingredients take a simplified form. In this case $\psi_0 = \sqrt{n_0}$ is constant and u, v are plane wave functions: $u = U(\mathbf{p}) \exp(i\mathbf{p} \cdot \mathbf{r})/\sqrt{V}$ and

$v = V(\mathbf{p}) \exp(i\mathbf{p} \cdot \mathbf{r})/\sqrt{V}$. After some straightforward algebra, we can rewrite the rate of the collective excitation propagating with momentum \mathbf{q} in the following way:

$$\gamma = -\omega\pi \int \frac{d\mathbf{p}_i}{(2\pi)^3} |B_{ki}|^2 \delta(E_k - E_i - \omega) \frac{\partial f(E_i)}{\partial E}, \quad (8)$$

where

$$B_{ki} = 2g\sqrt{n_0} \{ [U(E_k)V(E_i) + V(E_k)V(E_i) + U(E_k)U(E_i)]U_{osc} + [V(E_k)V(E_i) + V(E_k)V(E_i) + U(E_k)U(E_i)]V_{osc} \}, \quad (9)$$

$$U^2(E) = 1 + V^2(E) = \frac{(E^2 + g^2 n_0^2)^{1/2} + E}{2E} \quad (10)$$

$$U(E)V(E) = -\frac{gn_0}{2E}$$

and

$$E(\mathbf{p}) = \left(\left(\frac{p^2}{2m} + gn_0 \right)^2 - g^2 n_0^2 \right)^{1/2}. \quad (11)$$

is the dispersion law of the elementary excitations in the so called Popov approximation [10,11], depending on the corresponding value of the momentum \mathbf{p} . In these equations $n_0 \equiv n_0(T)$ is the condensate density at temperature T and the momenta \mathbf{p} of the i -th and k -th excitations satisfy the condition $\mathbf{p}_k = \mathbf{p}_i + \mathbf{q}$, following from momentum conservation.

A further simplification is obtained if one considers the damping for the low frequency excitations, i. e. for phonons ($\mathbf{q} \rightarrow 0$). In this case, one has

$$U_{osc}(\omega) \approx \left(\frac{gn_0}{2\omega} \right)^{1/2} + \frac{1}{2} \left(\frac{\omega}{2gn_0} \right)^{1/2} \quad (12)$$

and

$$V_{osc}(\omega) \approx -\left(\frac{gn_0}{2\omega} \right)^{1/2} + \frac{1}{2} \left(\frac{\omega}{2gn_0} \right)^{1/2}. \quad (13)$$

In the same limit $\mathbf{q} \rightarrow 0$, using momentum conservation, one can write $E_k - E_i \approx v_g \omega \cos(\theta)/c$, where θ is the angle between \mathbf{p}_i and \mathbf{q} , v_g is the group velocity of the i -th excitation and c is the velocity of sound. After integration with respect to θ and some straightforward algebra one finds the useful result

$$\gamma = -c\omega \int \frac{p^2 dp}{4\pi v_g} |C|^2 \frac{\partial f(E)}{\partial E}, \quad (14)$$

where

$$C(E) = \sqrt{2g} [U^2(E) + V^2(E) + U(E)V(E) + E \frac{\partial U^2(E)}{\partial E}]. \quad (15)$$

Result (14) permits to explore explicitly the T -dependence of Landau damping. At low T ($T \ll \mu$, where $\mu = gn_0(0)$ is the $T = 0$ value of the chemical potential), one finds the result

$$\gamma = \frac{27\pi}{16} \frac{\omega \rho_n}{\rho}, \quad (16)$$

where ρ_n is the normal part density of the phonon gas [12], §23:

$$\rho_n = \int \frac{p^4 dp}{6\pi^2} \left(-\frac{\partial f(E)}{\partial E} \right) = \frac{2\pi^2 T^4}{45\hbar^3 c^5}. \quad (17)$$

Eq. (16) has been first derived by P.Hohenberg and P.Martin [8]. This equation can be also obtained following the method of [13]. (See [5], Problem to §77.)

A second important regime occurs at relatively high T ($T \gg \mu$). Usually at such temperatures the thermodynamic quantities are determined by the excitations with $E \sim T$. In the integral (14) however the relevant excitations turn out to have energies $E \sim \mu \sim gn_0 \ll T$, so that one can use the "Rayleigh-Jeans" limit of the distribution function, $f(E) \approx T/E$. As a consequence the resulting dependence of γ is linear in T and integration of (14) gives the result

$$\gamma = \frac{3\pi}{8} \frac{Taq}{\hbar^2} \quad (18)$$

where we have used the expression $c^2 = gn_0/m$ for the sound velocity. This regime has been previously investigated by Szepfalusy and Kondor [9] and more recently by Hua Shi [14]. Our result coincides with the one obtained in [14] but the numerical coefficient slightly differs from the one of [9]. It is worth noting that result (18) does not depend on the value of the condensate fraction n_0 . The above "high" T regime has been recently employed in [7] to provide a quantitative estimate of the width of the collective excitations of a dilute Bose gas trapped by a harmonic potential. Estimating the momentum q of the excitation as $\hbar\omega/c$ where ω is the frequency of the excitation and $c = (\hbar/m)[4\pi an_0(r=0)]^{1/2}$ is the value of the sound velocity calculated in the center of the trap, the authors of [7] have obtained values for γ in semi-quantitative agreement with the experimental findings [1]. This agreement suggests that Landau damping could represent an important mechanism to explain the decay of elementary excitations of trapped Bose gases. Notice however that in a non uniform system the elementary excitations cannot be described in terms of plane wave functions and that Landau damping should be consequently calculated starting directly from the general equation (7), which involves matrix elements and eigenvalues relative to the eigenstates of the system. The presence of the trapping potential can influence the damping mechanism in a deep way. For example the dipole oscillation does not exhibit any damping in the presence of harmonic trapping.

It is finally interesting to discuss the validity of the "high" temperature expansion (18). To this purpose it is useful to write the ratio γ/ω in the form (see Eq. (14))

$$\frac{\gamma}{\omega} = (a^3 n_0(T))^{1/2} F(\tau) \quad (19)$$

where $\tau = T/mc^2(T)$ is a dimensionless variable and the function F is given by

$$F(\tau) = \frac{4\sqrt{\pi}}{\tau} \int_0^\infty dx \left(1 - \frac{1}{2u} - \frac{1}{2u^2}\right)^2 (e^{x/2\tau} - e^{-x/2\tau})^{-2} \quad (20)$$

with $u(x) = (1+x^2)^{1/2}$. For large τ (high temperatures) the function F takes the asymptotic value $F \rightarrow \frac{3}{4}\pi^{3/2}\tau$, consistent with result (18). In Fig. 1 we plot the function F versus τ . One can see that F approaches the asymptotic linear law rather slowly. For example for temperatures $kT \sim mc^2$ ($\tau \sim 1$) it differs from it by a factor 2. This suggests that the linear approximation (18) should be employed with care. In fact in the available traps [1] the value of mc^2 , with the sound velocity calculated in the center of the trap, corresponds to about $0.4T_c$ [11], where T_c is the critical temperature for Bose-Einstein condensation in the presence of the trapping potential, and consequently the condition $\tau \gg 1$ is never reached below T_c . This should be taken into account for quantitative estimates of the width using the present formalism.

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After completing the present work we have been informed of a similar analysis of Landau damping [15] carried out in dilute Bose gases using semiclassical theory.

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FIG. 1. Function F plotted as a function of τ (full line). The linear behavior $F \rightarrow (3/4)\pi^{3/2}\tau$ is also reported (dashed line).